Criterion for the unconditional basicity of the root functions related to the second-order differential operator with involution

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Let L be any operator related to the operation of the form $Lu \equiv -u''(x) + \alpha u''(-x) + q(x)u(x) + q_v(x)u(v(x))\,,\ -1 < x < 1$, (1)
and defined on a dense in $L_2(-1,1)$ domain $D(L)$. The operation (1) contains the argument's transform $v_0(x) = -x$, in its main term. This transform is called a simple involution (reflection) of the segment $[-1;1]$ and also n arbitrary involution $v(x)$ in the lower term. The parameter α in (1) belongs to (-1;1), the coefficients $q(x)$ and $q_v(x)$ are arbitrary and complex-valued integrable on $[-1;1]$ functions, the involution $\nu(x)$ is any absolutely continuous function which has an essentially bounded derivative on $[-1;1]$.

The particular form of the domain $D(L)$ will not be refined below; the operator L can be generated by the functional-differential operation (3.1), for example, with some boundary conditions on the segment $[-1;1]$. We only assume that the domain $D(L)$ contains only functions that, together with their first derivatives, are absolutely continuous on the interval $(-1;1)$, while the root functions of the operator L are considered as regular solutions of the corresponding equations with a spectral parameter.

Following Il'in [1], an eigenfunction (or a root function of the zero order) $u(x)$, that corresponds to the operator (3.1) and an eigenvalue $\lambda \in \mathbb{C}$ is defined as an arbitrary non trivial solution of the equation $Lu = \lambda u$. Here and throughout, a regular solution of the equation $Lu = f$ with a given right-hand side $f \in L_1(-1,1)$ is understood to be an arbitrary function $u(x)$ from the class $W_1^2(-1,1) \cap L_2(-1,1)$, that satisfies this equation almost everywhere on $(-1;1)$.

Let $\tilde{u}(x)-$ be a root function of order (k $-$ 1) $(k \geq 1)$, corresponding to an eigenvalue λ. Then the regular solution of the equation $Lu = \lambda u - \tilde{u}$ will be called its counterpart root (associated) function of order k.

For each eigenvalue $\lambda \in C$, we have there by defined a chain of root functions $u_k(x;\lambda)\,,\ k \geq 0$ that satisfy the relations

$$Lu_k(x;\lambda) = \lambda u_k(x;\lambda) - sgnk \cdot u_{k-1}(x;\lambda)\,, \qquad (2)$$

moreover, $u_0(x;\lambda) \not\equiv 0$ on $(-1;1)$.

Any count able set $\Lambda = \{\lambda\} \subset \mathbb{C}$ defines the system of root functions $U = \{u_k(x;\lambda)\,|k = 0, ..., m(\lambda)\,, \lambda \in \Lambda\}$; here then on negative integer $m(\lambda)$ will be called the rank of the corresponding eigenfunction $u_0(x;\lambda)$.

Left the system U satisfy the following conditions A:

A1) the system U is complete and minimal in $L_2(-1,1)$;
A2) a system V that is biorthogonally adjoint to U consists of root functions $\upsilon_l(x;\lambda^*)$, $l = 0,...,m(\lambda^*)$, $\lambda^* \in \overline{\Lambda}$, $m(\lambda^*) = m(\lambda)$, (in the above-defined sense) of the formal adjoint operation

$$L^*\upsilon = -\upsilon''(x) + \alpha\upsilon''(-x) + \overline{q(x)}\,\upsilon(x) - \upsilon'(x)\,\overline{q_v(v(x))}\upsilon(v(x)), \qquad (3)$$

and the relation $\left(u_k(\cdot;\lambda), \upsilon_{m(\lambda)-l}(\cdot;\lambda^*)\right) = 1$ is valid if and only if $k = l$ and $\lambda^* = \overline{\lambda}$; while in the remaining cases the inner product on the left-hand side in relation (3.4) is zero;

A3) the ranks of the eigenfunctions are uniformly bounded: $\sup\limits_{\lambda\in\Lambda} m(\lambda) < \infty$

and the condition that the set Λ belongs to the Carleman parabole is satisfied $\sup\limits_{\lambda\in\Lambda}\left|\mathrm{Im}\sqrt{\lambda}\right| < \infty$;

A4) the following uniform estimate of the "sum of units" is valid:

$$\sup_{\beta\geq 1} \sum_{\lambda\in\Lambda:\left|\mathrm{Re}\sqrt{\lambda-\beta}\right|\leq 1} 1 < \infty.$$

Theorem 1. Let the conditions 1-4 be satisfied and let the involution $\nu(x)$ occurring in (1) be an arbitrary continuous function with the derivative that is essentially bounded on the segment $[-1,1]$. Then each of the systems U and V of root functions of the operators (1) and (3), respectively, forms an unconditional basis in $L_2(-1,1)$ if and only if the uniform estimate of the product of norms $\|u_k(\cdot;\lambda)\|_2 \cdot \left\|\upsilon_{m(\lambda)-k}\left(\cdot;\bar{\lambda}\right)\right\|_2 \leq M$holds for all $\mathrm{k} = 0,\ldots,\mathrm{m}(\lambda)$and $\lambda \in \Lambda$The main theorem is complemented with the proof of the necessity of condition A4 in the case where the involution $\nu(x)$in the operator (1) is a reflection.

Theorem 2. Let the condition A3 be satisfied and, in addition, let $\nu(\mathrm{x}) = -\mathrm{x}$ If the system of root functions U that is normed in $\mathrm{L}_2(-1,1)$ possesses the Bessel property, then the uniform estimate of the "sum of units" A4 is valid.

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References

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